STATISTICAL MECHANICS
of
SELF-AVOIDING CRUMPLED MANIFOLDS

Part II

Bertrand Duplantier*

Service de Physique Théorique**
CEA/ Saclay
F-91191 Gif-sur-Yvette, France

Abstract†

We consider a model of a D-dimensional tethered manifold interacting by excluded volume in $\mathbb{R}^d$ with a single point. Use of intrinsic distance geometry provides a rigorous definition of the analytic continuation of the perturbative expansion for arbitrary D, $0 < D < 2$. Its one-loop renormalizability is first established by direct resummation. A renormalization operation $\mathbf{R}$ is then described, which ensures renormalizability to all orders. The similar question of the renormalizability of the self-avoiding manifold (SAM) Edwards model is then considered, first at one-loop, then to all orders. We describe a short-distance multi-local operator product expansion, which extends methods of local field theories to a large class of models with non-local singular interactions. It vindicates the direct renormalization method used earlier in part **I** of these lectures, as well as the corresponding scaling laws.

* e-mail: bertrand@spht.saclay.cea.fr

** Laboratoire de la Direction des Sciences de la Matière du Commissariat à l'Énergie Atomique, URA CNRS 2306.

† Published in *Statistical Mechanics of Membranes and Surfaces*, Second Edition, D. R. Nelson, T. Piran, and S. Weinberg eds., World Scientific, Singapore (2004).

1. INTERACTING MANIFOLD RENORMALIZATION: A BRIEF HISTORY

As can be seen in the set of lectures in this volume, which presents an extended version of [1], the statistical mechanics of random surfaces and membranes, or more generally of extended objects, poses fundamental problems. The study of *polymerized* membranes, which are generalizations of linear polymers [2,3] to two-dimensionally connected networks, is emphasized, with a number of possible experimental realizations [4,5,6,7,8], or numerical simulations [9,10]. From a theoretical point of view, a clear challenge in the late eighties was to understand self-avoidance (SA) effects in membranes.

The model proposed[1] in [11,12] aimed to incorporate the advances made in polymer theory by renormalization group (RG) methods into the field of polymerized, or tethered, membranes. As we saw in part **I** of these lectures, these extended objects, *a priori* two-dimensional in nature, are generalized for theoretical purposes to intrinsically D–*dimensional manifolds* with internal points $x \in \mathbb{R}^D$, embedded in external d-dimensional space with position vector $\vec{\mathbf{r}}(x) \in \mathbb{R}^d$. The associated continuum Hamiltonian $\mathcal{H}$ generalizes that of Edwards for polymers [2]:

$$\beta\mathcal{H} = \frac{1}{2}\int d^D x \left(\nabla_x \vec{\mathbf{r}}(x)\right)^2 + \frac{b}{2}\int d^D x \int d^D x' \; \delta^d\big(\vec{\mathbf{r}}(x) - \vec{\mathbf{r}}(x')\big) \; , \tag{1.1}$$

with an elastic Gaussian term and a self-avoidance two-body δ-potential with interaction parameter $b > 0$. For $0 < D < 2$, the Gaussian manifold ($b = 0$) is *crumpled* with a Gaussian size exponent

$$\nu_0 = \frac{2-D}{2}, \tag{1.2}$$

and a finite Hausdorff dimension

$$d_H = D/\nu_0 = 2D/(2-D); \tag{1.3}$$

the finiteness of the upper critical dimension $d^\star = 2d_H$ for the SA-interaction allows an ε-expansion about $d^\star$ [11–13]:

$$\varepsilon = 4D - 2\nu_0 d \tag{1.4}$$

[1] R.C. Ball was actually the first to propose, while a postdoc in Saclay in 1981, the extension of the Edwards model to D-manifolds, with the aim, at that time, to better understand polymers! (unpublished).

performed via the direct renormalization method adapted from that of des Cloizeaux in polymer theory [14], as we explained in part **I**.

Only the polymer case, with an *integer* internal dimension $D = 1$, can be mapped, following de Gennes [15], onto a standard field theory, namely a $(\mathbf{\Phi^2}(\vec{\mathbf{r}}))^2$ theory for an n-component field $\mathbf{\Phi}(\vec{\mathbf{r}})$ in external d-dimensional space, with $n \to 0$ components. This is instrumental in showing that the direct renormalization method for polymers is mathematically sound [16], and equivalent to rigorous renormalization schemes in standard local field theory, such as the Bogoliubov–Parasiuk–Hepp–Zimmermann (BPHZ) construction [17]. For manifold theory, we have to deal with *non-integer* internal dimensions D, $D \neq 1$, and no such mapping exists. Therefore, two outstanding problems remained in the theory of interacting manifolds: (a) the mathematical meaning of a *continuous* internal dimension D; (b) the actual *renormalizability* of the perturbative expansion of a manifold model like (1.1), implying the scaling behavior expected on physical grounds.

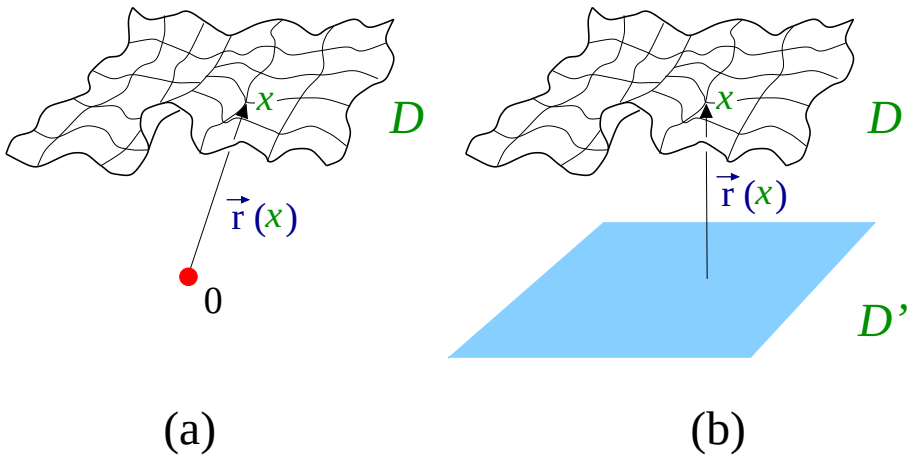

Fig. 1: *(a) A D-manifold interacting with an impurity located at point* **0** *in* $\mathbb{R}^d$*; (b) interaction with an Euclidean hyperplane of dimension D' in* $\mathbb{R}^{d'}$*, with* $d' = d + D'$.

In [18], a simpler model was proposed, of a crumpled manifold interacting by excluded volume with a fixed Euclidean subspace of $\mathbb{R}^d$ [19]. The simplified model Hamiltonian introduced there reads:

$$\beta\mathcal{H} = \frac{1}{2}\int d^D x \left(\nabla_x \vec{\mathbf{r}}(x)\right)^2 + b\int d^D x\ \delta^d\big(\vec{\mathbf{r}}(x)\big)\ , \tag{1.5}$$

with a pointwise interaction of the Gaussian manifold with an impurity located at the origin (Fig. 1a). Note that this Hamiltonian also represents interactions of a fluctuating

(possibly directed) manifold with a nonfluctuating D'- Euclidean hyperplane of $\mathbb{R}^{d+D'}$, $\vec{\mathbf{r}}$ then standing for the coordinates transverse to this subspace (Fig. 1b). The excluded volume case ($b > 0$) parallels that of the Edwards model (1.1) for SA-manifolds, while an attractive interaction ($b < 0$) is also possible, describing pinning phenomena. The (naive) dimensions of $\vec{\mathbf{r}}$ and b are respectively $[\vec{\mathbf{r}}] = [x^\nu]$ with a Gaussian size exponent

$$\nu \equiv (2 - D)/2, \tag{1.6}$$

and $[b] = [x^{-\varepsilon}]$ with

$$\varepsilon \equiv D - \nu d. \tag{1.7}$$

For fixed D and ν, the parameter d (or equivalently ε) controls the relevance of the interaction, with the exclusion of a point only effective for $d \leq d^\star = D/\nu$. Note that in this model the size exponent ν is not modified by the local interaction and stays equal to its Gaussian value (1.6), whereas the correlation functions obey (non-Gaussian) universal scaling laws.

For $D = 1$, the model is exactly solvable [18]. For $D \neq 1$, the direct resummation of leading divergences of the perturbation series is possible for model (1.5) and indeed validates *one-loop* renormalization [18]. This result was also extended to the Edwards model (1.1) itself [20].

A study to all orders of the interaction model (1.5) was later performed in [21], [22]. A mathematical construction of the D-dimensional internal measure $d^D x$ via distance geometry within the elastic manifold was given, with expressions for manifold Feynman integrals which generalize the α-parameter representation of field theory. In the case of the manifold model of [18], the essential properties which make it *renormalizable to all orders* by a renormalization of the coupling constant were established. This led to a direct construction of a renormalization operation, generalizing the BPHZ construction to manifolds (see also [23] [24].)

Later, the full Edwards model of self-avoiding manifolds (1.1) was studied by the same methods, and its renormalizability established to all orders [25], [26]. Effective calculations to second order in ε ("two-loop" order) were performed in [27]. The large order behavior of the Edwards model (1.1) was finally studied in [28].

The aim of part **II** of these notes is to review some of these developments.

2. MANIFOLD MODEL WITH LOCAL δ INTERACTION

2.1. Perturbative expansion

In this chapter, we study the statistical mechanics of the simplified model Hamiltonian (1.5). The model is described by its (connected) partition function

$$\mathcal{Z} = \mathcal{V}^{-1} \int \mathcal{D}[\vec{\mathbf{r}}] \exp(-\beta \mathcal{H}) \tag{2.1}$$

(here $\mathcal{V}$ is the internal volume of the manifold) and, for instance, by its one-point vertex function

$$\mathcal{Z}^{(0)}(\vec{\mathbf{k}})/\mathcal{Z} = \int d^D x_0 \ \langle e^{i\vec{\mathbf{k}}\cdot\vec{\mathbf{r}}(x_0)} \rangle, \tag{2.2}$$

where the (connected) average $\langle \cdots \rangle$ is performed with (1.5):

$$\mathcal{Z}^{(0)}(\vec{\mathbf{k}}) = \mathcal{V}^{-1} \int \mathcal{D}[\vec{\mathbf{r}}] \exp(-\beta \mathcal{H}) \int d^D x_0 \ e^{i\vec{\mathbf{k}}\cdot\vec{\mathbf{r}}(x_0)}. \tag{2.3}$$

These functions are all formally defined via their perturbative expansions in the coupling constant b:

$$\mathcal{Z} = \sum_{N=1}^{\infty} \frac{(-b)^N}{N!} \mathcal{Z}_N, \tag{2.4}$$

with a similar equation for $\mathcal{Z}^{(0)}$ with coefficients $\mathcal{Z}_N^{(0)}$:

$$\mathcal{Z}^{(0)}(\vec{\mathbf{k}}) = \sum_{N=1}^{\infty} \frac{(-b)^N}{N!} \mathcal{Z}_N^{(0)}(\vec{\mathbf{k}}). \tag{2.5}$$

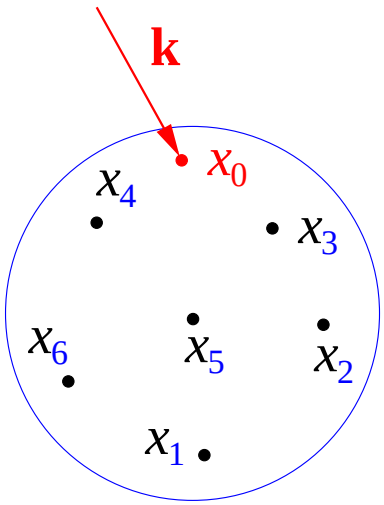

Fig. 2: *Interaction points x_i; insertion point x_0 for the external momentum $\vec{\mathbf{k}}$.*

$\mathcal{Z}_N$ has the path integral representation

$$\mathcal{Z}_N = \frac{1}{\mathcal{V}} \int d\mathcal{P}_0 \left[\int d^D x \; \delta^d\big(\vec{\mathbf{r}}(x)\big) \right]^N \tag{2.6}$$

where the Gaussian path measure is

$$d\mathcal{P}_0 = \mathcal{D}\vec{\mathbf{r}}(x) \exp(-\beta \mathcal{H}_0) \tag{2.7}$$

with

$$\beta \mathcal{H}_0 = \frac{1}{2} \int d^D x \left(\nabla_x \vec{\mathbf{r}}(x) \right)^2. \tag{2.8}$$

There is no translational invariance in this theory, since the origin is selected by the presence of the impurity. The measure $d\mathcal{P}_0$ thus *includes* integration over global translations of the manifold in $\mathbb{R}^d$. The first term is then simply $\mathcal{Z}_1 \equiv 1$, so that

$$\mathcal{Z} = -b + \mathcal{O}(b^2). \tag{2.9}$$

The term of order N, $\mathcal{Z}_N$, is a Gaussian average involving N interaction points x_i (Fig. 2):

$$\mathcal{Z}_N = \frac{1}{\mathcal{V}} \int d\mathcal{P}_0 \int \prod_{i=1}^N d^D x_i \; \prod_{i=1}^N \delta^d\big(\vec{\mathbf{r}}(x_i)\big). \tag{2.10}$$

By Fourier transforming the distribution in d-space

$$\delta^d\big(\vec{\mathbf{r}}(x)\big) = \int \frac{d^d \vec{\mathbf{k}}}{(2\pi)^d} \exp\left(i \vec{\mathbf{k}}.\vec{\mathbf{r}} \right),$$

one gets

$$\mathcal{Z}_N = \frac{1}{\mathcal{V}} \int \prod_{i=1}^N d^D x_i \int \prod_{i=1}^N \frac{d^d \vec{\mathbf{k}}_i}{(2\pi)^d} \int d\mathcal{P}_0 \exp\left[i \sum_{i=1}^N \vec{\mathbf{k}}_i.\vec{\mathbf{r}}_i \right]. \tag{2.11}$$

For a Gaussian manifold with weight (2.7) (2.8) we have:

$$\int d\mathcal{P}_0 \exp\left[i \sum_{i=1}^N \vec{\mathbf{k}}_i.\vec{\mathbf{r}}_i \right] = (2\pi)^d \delta^d \left(\sum_{i=1}^N \vec{\mathbf{k}}_i \right) \exp\left[-\frac{1}{2} \sum_{i,j=1}^N \vec{\mathbf{k}}_i.\vec{\mathbf{k}}_j G(x_i - x_j) \right]. \tag{2.12}$$

This Gaussian manifold average is expressed solely in terms of the Green function

$$G(x-y) = -\frac{1}{2} A_D |x-y|^{2\nu}, \tag{2.13}$$

solution[2] of

$$-\Delta_x G(x-y) = \delta^D(x-y), \tag{2.14}$$

with $2\nu = 2 - D$, and A_D a normalization:

$$A_D = [S_D(2-D)/2]^{-1} = [S_D \nu]^{-1}\,, \tag{2.15}$$

where S_D is the area of the unit sphere in D dimensions

$$S_D = \frac{2\pi^{D/2}}{\Gamma(D)}. \tag{2.16}$$

In the following, it is important to preserve the condition $0 < \nu < 1$ (*i.e.*, $0 < D < 2$), corresponding to the actual case of a crumpled manifold, where $(-G)$ is positive and ultraviolet (UV) finite.

Performing finally the Gaussian integral over the $N-1$ independent real variables $\vec{\mathbf{k}}_i$, $(i = 1, \cdots, N-1)$ yields [18]:

$$\mathcal{Z}_N \;= \mathcal{V}^{-1}\,(2\pi)^{-(N-1)d/2} \int \prod_{i=1}^{N} d^D x_i \,\left(\det\left[\Pi_{ij}\right]_{1\le i,j\le N-1}\right)^{-\frac{d}{2}}, \tag{2.17}$$

where the matrix $[\Pi_{ij}]$ is simply defined as

$$\Pi_{ij} \;\equiv\; G(x_i - x_j) - G(x_i - x_N) - G(x_j - x_N), \tag{2.18}$$

with respect to the reference point x_N, the permutation symmetry between the N points being restored in the determinant.

The integral representation of $\mathcal{Z}_N^{(0)}$ is obtained from that of $\mathcal{Z}_N$ by multiplying the integrand in (2.17) by $\exp(-\frac{1}{2}\vec{\mathbf{k}}^2\Delta^{(0)})$ with :

$$\Delta^{(0)} \equiv \frac{\det[\Pi_{ij}]_{0\le i,j\le N-1}}{\det[\Pi_{ij}]_{1\le i,j\le N-1}}\,, \tag{2.19}$$

and integrating over one more position, x_0, (Fig. 2):

$$\mathcal{Z}_N^{(0)}(\vec{\mathbf{k}}) \;= \mathcal{V}^{-1}\,(2\pi)^{-(N-1)d/2} \int \prod_{i=0}^{N} d^D x_i \,\exp\left(-\frac{1}{2}\vec{\mathbf{k}}^2\Delta^{(0)}\right)\,\left(\det\left[\Pi_{ij}\right]_{1\le i,j\le N-1}\right)^{-\frac{d}{2}}. \tag{2.20}$$

Notice that the first order term ($N = 1$) specializes to:

$$\mathcal{Z}_1^{(0)}(\vec{\mathbf{k}}) \;= \mathcal{V}^{-1} \int_{\mathcal{V}\times\mathcal{V}} d^D x_0\, d^D x_1 \exp\left(-\frac{1}{2}\vec{\mathbf{k}}^2\Pi_{01}\right). \tag{2.21}$$

The resulting expressions are quite similar to those for the Edwards manifold model [20].

[2] In part **I** we used the notation $G(x-y) \equiv A_D|x-y|^{2\nu}$ for the (positive) solution of the slightly different equation $\Delta_x G(x-y) = 2\delta^D(x-y)$, while hereafter in **II** we shall use the proper Newton-Coulomb potential (2.13), in view of the underlying electrostatic representation.

2.2. **Second virial coefficient**

In this section, we imagine the manifold to be of finite internal volume $\mathcal{V} = X^D$, and define two dimensionless interaction coefficients, the excluded volume parameter z, and the second virial coefficient g, as

$$z = (2\pi A_D)^{-d/2} b X^{D-(2-D)d/2}, \tag{2.22}$$

$$g = (2\pi A_D)^{-d/2} (-\mathcal{Z}) X^{D-(2-D)d/2}. \tag{2.23}$$

Because of (2.9) , the perturbative expansion of the full interaction parameter g starts as:

$$g = z + \mathcal{O}(z^2). \tag{2.24}$$

More precisely we have:

$$g = \sum_{N=1}^{\infty} (-1)^{N-1} z^N I_N \tag{2.25}$$

where we have set

$$\frac{1}{N!} \mathcal{Z}_N \equiv (2\pi A_D)^{-(N-1)d/2} X^{(N-1)\varepsilon} I_N \tag{2.26}$$

in order to get rid of cumbersome factors. Now the dimensionless integral I_N is

$$I_N = \frac{1}{N!} \int_{\mathcal{V}'} \prod_{i=1}^{N} d^D x_i \ (\det \mathbf{D})^{-d/2}, \tag{2.27}$$

with integrations over *rescaled* coordinates, in a unit internal volume $\mathcal{V}' = X^{-D}\mathcal{V} = 1$; $\mathbf{D}$ is the symmetric $(N-1) \times (N-1)$ matrix with elements $(1 \leq i, j \leq N-1)$

$$\begin{aligned} D_{ii} &= |x_{iN}|^{2-D} \\ D_{ij} &= \frac{1}{2} \left(|x_{iN}|^{2-D} + |x_{jN}|^{2-D} - |x_{ij}|^{2-D} \right), \end{aligned} \tag{2.28}$$

where we set $x_{ij} \equiv x_i - x_j$.

2.3. **Resummation of leading divergences**

In this section we analyse the leading divergence of each I_N for $\varepsilon = D - (2-D)d/2 = D - \nu d > 0$. We have $I_1 = 1$, and

$$I_2 = \frac{1}{2} \int_{\mathcal{V}' \times \mathcal{V}'} d^D x_1 \ d^D x_2 \ |x_1 - x_2|^{-(2-D)d/2}. \tag{2.29}$$

We are interested in evaluating the pole at $\varepsilon = 0$. It is easily extracted as [18]

$$I_2 \simeq \frac{1}{2} \int_{\mathcal{V}'} d^D x_1 \int_0^1 S_D \; dy \; y^{-1+\varepsilon} = \frac{S_D}{2\varepsilon}, \tag{2.30}$$

where[3] $y = |x_1 - x_2|$.

The structure of divergences of the generic term I_N will be studied in detail in the next sections. They will be shown to be only *local* divergences, obtained by letting any interaction point subset coalesce. Here, the leading divergence is evaluated as follows.

The determinant in (2.27) is symmetrical with respect to the N points, so we can, for a given $i \in \{1, \cdots, N-1\}$, and without loss of generality, consider the "Hepp sector" $x_i \to x_N$, hence $\rho \equiv |x_{iN}| \to 0$. We then have $D_{ii} = |x_{iN}|^{2-D}$, while for any other $j \in \{1, \cdots, N-1\}$, $j \neq i$,

$$D_{ij} \simeq \frac{1}{2} \left(|x_{iN}|^{2-D} - x_{iN}.\nabla |x_{jN}|^{2-D} + \mathcal{O}(\rho^2) \right).$$

Using $\nu = (2-D)/2$ and the notation $\delta \equiv \min(\nu, 1-\nu)$, we can write the leading term of this equation, which depends on the position of D, $0 < D < 2$, with respect to 1, as

$$D_{ij} = |x_{iN}|^{\nu} \times \mathcal{O}\left(\rho^{\delta}\right).$$

When expanding the determinant $\det \mathbf{D}$ with respect to column i and line i, we encounter either the diagonal term $D_{ii} = |x_{iN}|^{2\nu} = \mathcal{O}(\rho^{2\nu})$, or non diagonal terms of type $D_{ij} D_{ik} = \mathcal{O}\left(\rho^{2\nu+2\delta}\right)$. Thus D_{ii} dominates and we can write in the sector $x_i \to x_N$

$$\det \mathbf{D} \simeq D_{ii} \times \det \mathbf{D}/i = |x_{iN}|^{2-D} \times \det \mathbf{D}/i, \tag{2.31}$$

where $\det \mathbf{D}/i$ is the reduced determinant of order $(N-2) \times (N-2)$, in which line i and column i have been removed, hence the point i itself. By symmetry, in any other sector $x_i \to x_j$, we have similarly

$$\det \mathbf{D} \simeq |x_{ij}|^{2-D} \times \det \mathbf{D}/i. \tag{2.32}$$

Among the $N(N-1)/2$ possible pairs (i,j) we define an arbitrary ordered set of $N-1$ pairs $\mathcal{P} = \{(i_\alpha, j_\alpha), \alpha = 1, \cdots, N-1\}$, such that the distances $|x_{i_\alpha} - x_{j_\alpha}| = y_\alpha \to 0$ define

[3] Note that the precise value of the upper limit for y, $y \lesssim 1$, is immaterial when evaluating the pole part.

a *sector* $y_1 \leq y_2 \leq \cdots \leq y_{N-1}$. In this limit, applying the rule (2.32) successively from $\alpha = 1$ to $N-1$ yields a determinant factorized as

$$\det \mathbf{D} \simeq \prod_{\alpha=1}^{N-1} y_\alpha^{2-D}.$$

The contribution of the sector $\mathcal{P}$ to the integral I_N is given by the iteration of (2.30):

$$\begin{aligned} I_{N|\mathcal{P}} &\simeq \frac{1}{N!} \prod_{\alpha=1}^{N-1} \left[S_D \int_0^{y_{\alpha+1}} dy_\alpha \; y_\alpha^{-1+\varepsilon} \right] \\ &= \frac{1}{N!} \frac{1}{(N-1)!} \left[\frac{S_D}{\varepsilon} \right]^{N-1}. \end{aligned} \tag{2.33}$$

The number of distinct sectors of $N-1$ ordered pairs $\mathcal{P}$ chosen among N points equals $N!(N-1)!/2^{N-1}$, whence the leading divergence of I_N:

$$I_N \simeq \left(\frac{S_D}{2\varepsilon} \right)^{N-1}. \tag{2.34}$$

At this order, the dimensionless excluded volume parameter g (2.23) thus reads

$$\begin{aligned} g &= \sum_{N=1}^{\infty} (-1)^{N-1} z^N I_N \simeq \sum_{N=1}^{\infty} (-1)^{N-1} z^N \left(\frac{S_D}{2\varepsilon} \right)^{N-1} \\ &= \frac{z}{1 + z\frac{S_D}{2\varepsilon}}. \end{aligned} \tag{2.35}$$

2.4. **Comparison to one-loop renormalization**

The Taylor-Laurent expansion of parameter g to first orders is obtained from (2.25) and (2.30)

$$g = z - z^2 I_2 + \cdots = z - z^2 \frac{S_D}{2\varepsilon} + \cdots. \tag{2.36}$$

It is associated with a Wilson function

$$\begin{aligned} W(g,\varepsilon) &= X \frac{\partial g}{\partial X} = \varepsilon z \frac{\partial g}{\partial z} \\ &= \varepsilon z - z^2 S_D + \cdots = \varepsilon g - g^2 \frac{S_D}{2} + \cdots. \end{aligned} \tag{2.37}$$

The fixed point g^* such that $W(g^*, \varepsilon) = 0$ is $g^* = 2\varepsilon/S_D$ and precisely corresponds to the limit of (2.35)

$$g(z \to +\infty) = \frac{2\varepsilon}{S_D} = g^*. \tag{2.38}$$

More interestingly, the (truncated) flow equation (2.37)

$$W(g,\varepsilon) = \varepsilon z \frac{\partial g}{\partial z} = \varepsilon g - g^2 \frac{S_D}{2}\,, \tag{2.39}$$

with boundary condition (2.24), has precisely the solution $g = z/(1 + z\frac{S_D}{2\varepsilon})$. So we see that the resummation (2.35) to all orders of leading divergences is exactly equivalent to the one-loop renormalization group equation, as displayed in (2.39). Thus the one-loop renormalizability of the manifold model has been directly established by direct resummation of the perturbation expansion [18].

This is confirmed by consideration of the vertex function (2.2). The same evaluation of (2.20) gives, after successive contractions of pairs of points in the determinants in (2.19), (2.20), the leading divergence:

$$\mathcal{Z}_N^{(0)}(\vec{\mathbf{k}}) \;= \frac{1}{\mathcal{V}}\,(2\pi A_D)^{-(N-1)d/2} X^{(N-1)\varepsilon} \left(\frac{S_D}{2\varepsilon}\right)^{N-1} N! \int_{\mathcal{V}\times\mathcal{V}} d^D x_0 \; d^D x_1 \; \exp\left\{-\frac{1}{2}\vec{\mathbf{k}}^2 \Pi_{01}\right\} \tag{2.40}$$

with the matrix element $\Pi_{01} = -2G(x_0 - x_1)$. The (connected) vertex function (2.3), (2.5) can thus be resummed at this order as

$$\begin{aligned} \mathcal{Z}^{(0)}(\vec{\mathbf{k}}) &= \sum_{N=1}^{\infty} \frac{(-b)^N}{N!}\, \mathcal{Z}_N^{(0)}(\vec{\mathbf{k}}) \\ &= \frac{-b}{1 + z\frac{S_D}{2\varepsilon}}\; \mathcal{V}^{-1} \int_{\mathcal{V}\times\mathcal{V}} d^D x_0 \; d^D x_1 \; \exp\left\{-\frac{1}{2}\vec{\mathbf{k}}^2 (-2G(x_0 - x_1))\right\}. \end{aligned} \tag{2.41}$$

Notice that, at *first order*, $\mathcal{Z}^{(0)}$ is determined from (2.21) as

$$\begin{aligned} \mathcal{Z}^{(0)}(\vec{\mathbf{k}}) &= -b\mathcal{Z}_1^{(0)}(\vec{\mathbf{k}}) + \mathcal{O}(b^2) \\ &= -b\; \mathcal{V}^{-1} \int_{\mathcal{V}\times\mathcal{V}} d^D x_0 \; d^D x_1 \; \exp\left\{-\frac{1}{2}\vec{\mathbf{k}}^2 (-2G(x_0 - x_1))\right\} + \mathcal{O}(b^2); \end{aligned} \tag{2.42}$$

therefore the resummation of leading divergences in (2.41) amounts exactly to replacing

$$b \to \frac{b}{1 + z\frac{S_D}{2\varepsilon}}$$

in the first order correlation function (2.42). Owing to (2.22), this is indeed equivalent to replacing the bare dimensionless interaction parameter z by the renormalized one $g = z/(1 + z\frac{S_D}{2\varepsilon})$, in complete agreement with (2.35) above.

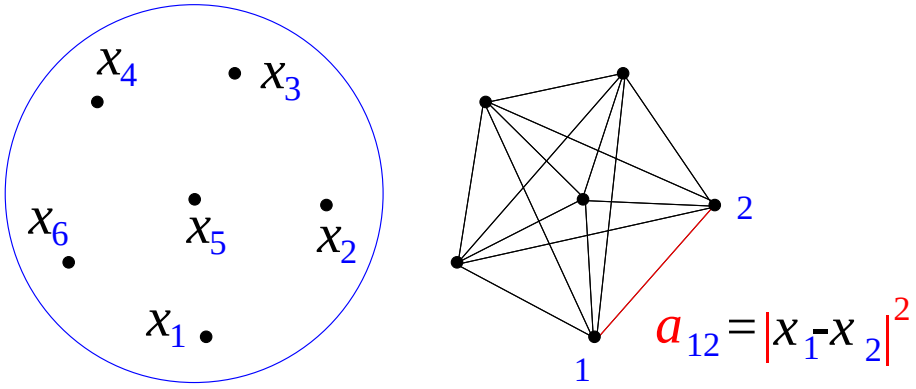

Fig. 3: *Passage from Euclidean coordinates x_i to the complete set of squared distances a_{ij}.*

2.5. Analytic continuation in D of the Euclidean measure

Integrals like (2.17) or (2.20), written with Cartesian coordinates, are *a priori* meaningful only for integer D. Up to now, we have only formally extended such integrals to non-integer dimensions. Actually, an analytic continuation in D can be performed by use of *distance geometry* [22]. The key idea is to substitute for the internal Euclidean coordinates x_i the set of all mutual (squared) distances $a_{ij} = (x_i - x_j)^2$ (Fig. 3).

This is possible for integrands invariant under the group of Euclidean motions (as in (2.17) and (2.20)). For N integration points, it also requires, *before* analytic continuation, D to be large enough, *i.e.*, $D \geq N-1$, such that the $N-1$ relative vectors spanning these points are linearly independent.

We define the graph $\mathcal{G}$ as the set $\mathcal{G} = \{1, \ldots, N\}$ labelling the interaction points. Vertices $i \in \mathcal{G}$ will be remnants of the original Euclidean points after analytic continuation, and index the squared distance matrix $[a_{ij}]$. The change of variables $\{x_i\}_{i\in\mathcal{G}} \to a \equiv [a_{ij}]_{\substack{i<j \\ i,j\in\mathcal{G}}}$ reads explicitly [22]:

$$\frac{1}{\mathcal{V}} \int_{\mathbb{R}^D} \prod_{i\in\mathcal{G}} d^D x_i \cdots = \int_{\mathcal{A}_\mathcal{G}} d\mu_\mathcal{G}^{(D)}(a) \cdots , \tag{2.43}$$

with the measure

$$d\mu_\mathcal{G}^{(D)}(a) \equiv \prod_{\substack{i<j \\ i,j\in\mathcal{G}}} da_{ij}\ \Omega_N^{(D)} \left(P_\mathcal{G}(a)\right)^{\frac{D-N}{2}} , \tag{2.44}$$

where $N = |\mathcal{G}|$, and

$$\Omega_N^{(D)} \equiv \prod_{K=0}^{N-2} \frac{S_{D-K}}{2^{K+1}} \tag{2.45}$$

($S_D = \frac{2\pi^{D/2}}{\Gamma(D/2)}$ is as before the volume of the unit sphere in $\mathbb{R}^D$), and

$$P_{\mathcal{G}}(a) \equiv \frac{(-1)^N}{2^{N-1}} \begin{vmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & 0 & a_{12} & \dots & a_{1N} \\ 1 & a_{12} & 0 & \dots & a_{2N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & a_{1N} & a_{2N} & \dots & 0 \end{vmatrix} . \tag{2.46}$$

The factor $\Omega_N^{(D)}$ (2.45) is the volume of the rotation group of the rigid simplex spanning the points x_i. The "Cayley-Menger determinant" [29] $P_{\mathcal{G}}(a)$ is proportional to the squared Euclidean volume of this simplex, a polynomial of degree $N-1$ in the a_{ij}. The set a of squared distances has to fulfill the triangular inequalities and their generalizations: $P_{\mathcal{K}}(a) \geq 0$ for all subgraphs $\mathcal{K} \subset \mathcal{G}$, which defines the domain of integration $\mathcal{A}_{\mathcal{G}}$ in (2.43).

For real $D > |\mathcal{G}|-2 = N-2$, $d\mu_{\mathcal{G}}^{(D)}(a)$ is a positive measure on $\mathcal{A}_{\mathcal{G}}$, analytic in D. It is remarkable that, as a distribution, it can be extended to $0 \leq D \leq |\mathcal{G}|-2$ [22]. For integer $D \leq |\mathcal{G}|-2$, although the change of variables from x_i to a_{ij} no longer exists, Eq.(2.44) still reconstructs the correct measure, concentrated on D-dimensional submanifolds of $\mathbb{R}^{N-1}$, *i.e.*, $P_{\mathcal{K}} = 0$ if $D \leq |\mathcal{K}|-2$ [22]. For example, when $D \to 1$ for $N = 3$ vertices, we have, denoting the distances $|ij| = \sqrt{a_{ij}}$:

$$\frac{d\mu_{\{1,2,3\}}^{(D\to 1)}(a)}{d|12|d|13|d|23|} = 2\,\delta\big(|12|+|23|-|13|\big) + 2\,\delta\big(|13|+|32|-|12|\big) + 2\,\delta\big(|21|+|13|-|23|\big), \tag{2.47}$$

which indeed describes the 6 possibilities for nested intervals in $\mathbb{R}$, with degeneracy factors 2 corresponding to the reversal of the orientation.

Another nice feature of this formalism is that *the interaction determinants in (2.17) and (2.19) are also Cayley-Menger determinants!* We have indeed

$$\det\left[\Pi_{ij}\right]_{1\leq i,j\leq N-1} = P_{\mathcal{G}}(a^\nu) \tag{2.48}$$

where $a^\nu \equiv [a_{ij}^\nu]_{\substack{i<j \\ i,j\in\mathcal{G}}}$ is obtained by simply raising each squared distance to the power ν. We arrive for (2.17) and (2.20) at the representation of "Feynman diagrams" in distance geometry:

$$\begin{aligned} \mathcal{Z}_N &= \int_{\mathcal{A}_{\mathcal{G}}} d\mu_{\mathcal{G}}^{(D)}\, I_{\mathcal{G}} \ , \quad I_{\mathcal{G}} = \big(P_{\mathcal{G}}(a^\nu)\big)^{-\frac{d}{2}} \\ \mathcal{Z}_N^{(0)}(\vec{\mathbf{k}}) &= \int_{\mathcal{A}_{\mathcal{G}\cup\{0\}}} d\mu_{\mathcal{G}\cup\{0\}}^{(D)}\, I_{\mathcal{G}}^{(0)}(\vec{\mathbf{k}}) \ , \\ I_{\mathcal{G}}^{(0)}(\vec{\mathbf{k}}) &= I_{\mathcal{G}}\ \exp\left(-\frac{1}{2}\vec{\mathbf{k}}^2 \frac{P_{\mathcal{G}\cup\{0\}}(a^\nu)}{P_{\mathcal{G}}(a^\nu)}\right), \end{aligned} \tag{2.49}$$

which are D-dimensional extensions of the Schwinger α-parameter representation. We now have to study the actual convergence of these integrals and, possibly, their renormalization.

2.6. **Analysis of divergences**

Large distance infrared (IR) divergences occur for manifolds of infinite size. One can keep a finite size, preserve symmetries and avoid boundary effects by choosing as a manifold the D-dimensional sphere $\mathcal{S}_D$ of radius R in $\mathbb{R}^{D+1}$. This amounts [22] in distance geometry to substituting for $P_{\mathcal{G}}(a)$ the "spherical" polynomial $P^{\mathcal{S}}_{\mathcal{G}}(a) \equiv P_{\mathcal{G}}(a) + \frac{1}{R^2}\det(-\frac{1}{2}a)$, the second term providing an IR cut-off, such that $a_{ij} \leq 4R^2$. In the following, this IR regularization will simply be ignored when dealing with short-distance properties, for which we can take $P^{\mathcal{S}}_{\mathcal{G}} \sim P_{\mathcal{G}}$. This was also the case when evaluating leading divergences in the sections above.

The complete description of the possible set of divergences is then obtained from the following theorem of distance geometry [29]:

Schoenberg's theorem. For $0 < \nu < 1$, *the set* $a^\nu = [a^\nu_{ij}]_{\substack{i<j \\ i,j\in\mathcal{G}}}$ *can be realized as the set of squared distances of a transformed simplex in* $\mathbb{R}^{N-1}$, *whose volume* $P_{\mathcal{G}}(a^\nu)$ *is positive, and vanishes if and only if at least one of the mutual original distances itself vanishes,* $a_{ij} = 0$.

This ensures that, as in field theory, the only source of divergences in $I_{\mathcal{G}}$ and $I^{(0)}_{\mathcal{G}}$ is at *short distances*. Whether these UV singularities are integrable or not will depend on whether the external space dimension $d < d^\star = D/\nu$ or $d > d^\star$.

2.7. **Factorizations**

The key to convergence and renormalization is the following short-distance *factorization* property of $P_{\mathcal{G}}(a^\nu)$. Let us consider a subgraph $\mathcal{P} \subset \mathcal{G}$, with at least two vertices, in which we distinguish an element, the *root* p of $\mathcal{P}$, and let us denote by $\mathcal{G}_{/\!/_p}\mathcal{P} \equiv (\mathcal{G}\setminus\mathcal{P})\cup\{p\}$ the subgraph obtained by replacing in $\mathcal{G}$ the whole subset $\mathcal{P}$ by its root p. In the original Euclidean formulation, the analysis of short-distance properties amounts to that of contractions of points x_i, labeled by such a subset $\mathcal{P}$, toward the point x_p, according to: $x_i(\rho) = x_p + \rho(x_i - x_p)$ if $i \in \mathcal{P}$, where $\rho \to 0^+$ is the dilation factor, and $x_i(\rho) = x_i$ if $i \notin \mathcal{P}$. This transformation has an immediate resultant in terms of mutual distances: $a_{ij} \to a_{ij}(\rho)$, depending on both $\mathcal{P}$ and p. Under this transformation, the interaction polynomial $P_{\mathcal{G}}(a^\nu)$ factorizes into [22]:

$$\begin{aligned} P_{\mathcal{G}}(a^\nu(\rho)) = P_{\mathcal{P}}(a^\nu(\rho))\, P_{\mathcal{G}_{/\!/_p}\mathcal{P}}(a^\nu) \\ \times \left\{1 + \mathcal{O}(\rho^{2\delta})\right\} \quad . \end{aligned} \tag{2.50}$$

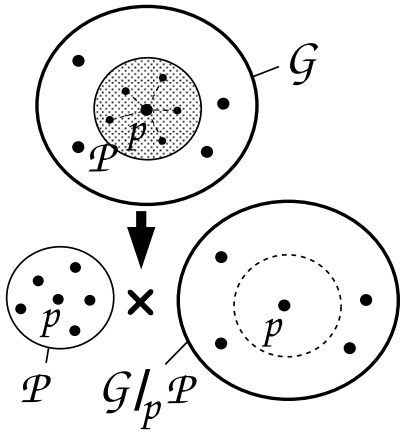

Fig. 4: *Factorization property (2.50).*

with $\delta = \min(\nu, 1-\nu) > 0$ and where, by homogeneity, $P_{\mathcal{P}}(a^\nu(\rho)) = \rho^{2\nu(|\mathcal{P}|-1)}\, P_{\mathcal{P}}(a^\nu)$. The geometrical interpretation of (2.50) is quite simple: the contribution of the set $\mathcal{G}$ splits into that of the contracting subgraph $\mathcal{P}$ multiplied by that of the whole set $\mathcal{G}$ where $\mathcal{P}$ has been replaced by its root p (Fig. 4), all correlation distances between these subsets being suppressed. The factorization property (2.50) is the generalization, to an arbitrary set $\mathcal{P}$ of contracting points, of the factorization encountered in (2.32) for the contraction of a pair of points. This is simply, in this interacting manifold model, the rigorous expression of an *operator product expansion* [22].

The factorization property (2.50) does not hold for $\nu = 1$, preventing a factorization of the measure (2.44) $d\mu_{\mathcal{G}}^{(D)}(a)$ itself. Still, the integral of the measure, when applied to a factorized integrand, does factorize as:

$$\int_{\mathcal{A}_{\mathcal{G}}} d\mu_{\mathcal{G}}^{(D)} \cdots = \int_{\mathcal{A}_{\mathcal{P}}} d\mu_{\mathcal{P}}^{(D)} \cdots \int_{\mathcal{A}_{(\mathcal{G}/_p\mathcal{P})}} d\mu_{(\mathcal{G}/_p\mathcal{P})}^{(D)} \cdots \ . \tag{2.51}$$

This fact, explicit for integer D with a readily factorized measure $\prod_i d^D x_i$, is preserved [22] by analytic continuation only after integration over relative distances between the two "complementary" subsets $\mathcal{P}$ and $\mathcal{G}/_p\mathcal{P}$.

2.8. **Renormalization**

A first consequence of factorizations (2.50) and (2.51) is the absolute convergence of $\mathcal{Z}_N$ and $\mathcal{Z}_N^{(0)}$ for $\varepsilon > 0$. Indeed, the superficial degree of divergence of $\mathcal{Z}_N$ (in distance

units) is $(N-1)\varepsilon$, as can be read from (2.49), already ensuring the superficial convergence when $\varepsilon > 0$. The above factorizations ensure that the superficial degree of divergence in $\mathcal{Z}_N$ or $\mathcal{Z}_N^{(0)}$ of any subgraph $\mathcal{P}$ of $\mathcal{G}$ is exactly that of $\mathcal{Z}_{|\mathcal{P}|}$ itself, *i.e.*, $(|\mathcal{P}|-1)\varepsilon > 0$. By recursion, this ensures the absolute convergence of the manifold Feynman integrals. A complete discussion has recourse to a generalized notion of Hepp sectors and is given in [22]. In the proof, it is convenient to first consider D large enough where $d\mu_{\mathcal{G}}^{(D)}$ is a non-singular measure, with a fixed ν considered as an independent variable $0 < \nu < 1$, and to then continue to $D = 2 - 2\nu$, $0 < D < 2$, corresponding to the physical case.

When $\varepsilon = 0$, the integrals giving $\mathcal{Z}_N$ and $\mathcal{Z}_N^{(0)}$ are (logarithmically) divergent. Another consequence of Eqs. (2.50) and (2.51) is thus the possibility to devise a renormalization operation $\mathbf{R}$, as follows. To each contracting rooted subgraph $(\mathcal{P}, p)$ of $\mathcal{G}$, we associate a Taylor operator $\mathfrak{T}_{(\mathcal{P},p)}$, performing on interaction integrands the exact factorization corresponding to (2.50):

$$\mathfrak{T}_{(\mathcal{P},p)} I_{\mathcal{G}}^{(0)} = I_{\mathcal{P}}\, I^{(0)}_{\mathcal{G}/\!\!/_p\mathcal{P}} \ , \tag{2.52}$$

and similarly $\mathfrak{T}_{(\mathcal{P},p)} I_{\mathcal{G}} = I_{\mathcal{P}}\, I_{\mathcal{G}/\!\!/_p\mathcal{P}}$. As in standard field theory [17], the subtraction renormalization operator $\mathbf{R}$ is then organized in terms of forests à la Zimmermann. In manifold theory, we define a *rooted forest* as a set of rooted subgraphs $(\mathcal{P}, p)$ such that any two subgraphs are either disjoint or nested, *i.e.*, never partially overlap. Each of these subgraphs in the forest will be contracted toward its root under the action (2.52) of the corresponding Taylor operator. When two subgraphs $\mathcal{P} \subset \mathcal{P}'$ are nested, the smallest one is contracted first toward its root p, the root p' of $\mathcal{P}'$ being itself attracted toward p if p' happened to be in $\mathcal{P}$. This hierarchical structure is anticipated by choosing the roots of the forest as *compatible*: in the case described above, if $p' \in \mathcal{P}$, then $p' \equiv p$. Finally, the renormalization operator is written as a sum over all such compatibly rooted forests of $\mathcal{G}$, denoted by $\mathcal{F}_{\oplus}$:

$$\mathbf{R} \ = \ \sum_{\mathcal{F}_{\oplus}} W(\mathcal{F}_{\oplus}) \left[\prod_{(\mathcal{P},p)\in\mathcal{F}_{\oplus}} \big(- \mathfrak{T}_{(\mathcal{P},p)} \big) \right] . \tag{2.53}$$

Here W is a necessary combinatorial weight associated with the degeneracy of compatible rootings, $W(\mathcal{F}_{\oplus}) = \prod_{\substack{p \text{ root} \\ \text{of } \mathcal{F}_{\oplus}}} 1/|\mathcal{P}(p)|$ with $\mathcal{P}(p)$ being the largest subgraph of the forest $\mathcal{F}_{\oplus}$ whose root is p. An important property is that, with compatible roots, the Taylor operators of a given forest now commute [22]. The renormalized amplitudes are defined as

$$\mathcal{Z}^{\mathbf{R}}{}_N^{(0)}(\vec{\mathbf{k}}) \ \equiv \ \int_{\mathcal{A}_{\mathcal{G}\cup\{0\}}} d\mu^{(D)}_{\mathcal{G}\cup\{0\}}\, \mathbf{R}\, [I_{\mathcal{G}}^{(0)}(\vec{\mathbf{k}})] \ . \tag{2.54}$$

The same operation $\mathbf{R}$ acting on $I_{\mathcal{G}}$ leads automatically by homogeneity to $\mathbf{R}\,[I_{\mathcal{G}}] = 0$ for $|\mathcal{G}| \geq 2$. We state the essential result that now *the renormalized Feynman integral* (2.54) *is convergent*: $\mathcal{Z}^{\mathbf{R}}{}_N^{(0)} < \infty$ for $\varepsilon = 0$. A complete proof of this renormalizability property is given in [22] the analysis being inspired from the direct proof by Bergère and Lam of the renormalizability in field theory of Feynman amplitudes in the α-representation [30].

The physical interpretation of the renormalized amplitude (2.54) and of (2.53) is simple. Equations (2.51) and (2.52) show that the substitution for the bare amplitudes (2.49) of the renormalized ones (2.54) amounts to a reorganization to all orders of the original perturbation series in b, leading to the remarkable identity:

$$\mathcal{Z}^{(0)}(\vec{\mathbf{k}}) \;=\; \sum_{N=1}^{\infty} \frac{(-b_{\mathbf{R}})^N}{N!}\, \mathcal{Z}^{\mathbf{R}}{}_N^{(0)}(\vec{\mathbf{k}})\;, \tag{2.55}$$

where the *renormalized* interaction parameter $b_{\mathbf{R}}$ is simply here (minus) the connected partition function

$$b_{\mathbf{R}} \equiv -\mathcal{Z}. \tag{2.56}$$

This actually extends to any vertex function, showing that the theory is made perturbatively finite (at $\varepsilon = 0$) by a full renormalization of the coupling constant b into $-\mathcal{Z}$ itself, in agreement with the definition of the second virial coefficient g (2.23) above. From this result, one establishes the existence to all orders of the Wilson function (2.37)

$$W(g, \varepsilon) = X \frac{\partial g}{\partial X}\Big|_b,$$

describing the scaling properties of the interacting manifold for ε close to zero, and which has a *finite limit* up to $\varepsilon = 0$ [22]. For $\varepsilon > 0$, an IR fixed point at $b > 0$ yields universal excluded volume exponents; for $\varepsilon < 0$, the associated UV fixed point at $b < 0$ describes a localization transition.

This demonstrated how to define an interacting manifold model with continuous internal dimension, by use of distance geometry, as a natural extension of the Schwinger representation for field theories. Furthermore, in the case of a pointwise interaction, the manifold model is indeed renormalizable to all orders. The main ingredients are Schoenberg's theorem of distance geometry, insuring that divergences occur only at short distances for (finite) manifolds, and the short-distance factorization of the generalized Feynman amplitudes. This provided probably the first example of a perturbative renormalization established for extended geometrical objects [22]. This opens the way to the renormalization theory of self-avoiding manifolds, which we now sketch.

3. SELF-AVOIDING MANIFOLDS & EDWARDS MODELS

3.1. Introduction

In this part, we concentrate on the renormalization theory of the model of tethered self-avoiding manifolds (SAM) [11,12], directly inspired by the Edwards model for polymers [2]:

$$\mathcal{H}/\mathrm{k_B}T \;=\; \frac{1}{2}\int d^Dx\,\big(\nabla_x\vec{\mathbf{r}}(x)\big)^2 + \frac{b}{2}\int d^Dx\int d^Dx'\;\delta^d\big(\vec{\mathbf{r}}(x)-\vec{\mathbf{r}}(x')\big)\;, \qquad (3.1)$$

with an elastic Gaussian term and a self-avoidance two-body δ-potential with excluded volume parameter $b>0$. Notice that in contrast with the local δ interaction model (1.5) studied in § 2, the interaction here is *non-local* in "manifold space" $\mathbb{R}^D$.

The finite upper critical dimension (u.c.d.) $d^\star$ for the SA interaction exists only for manifolds with a continuous internal dimension $0<D<2$. For $D\to 2$, $d^\star\to+\infty$. Phantom manifolds ($b=0$) are *crumpled* with a finite Hausdorff dimension $d_H=2D/(2-D)$, and $d^\star=2d_H$. The ε-expansion about $d^\star$ performed in [11,12,13], and described in part **I** above, was directly inspired by the des Cloizeaux *direct renormalization* (DR) method in polymer theory [14]. But the issue of the consistency of the DR method remained unanswered, since for $D\neq 1$, model (3.1) cannot be mapped onto a standard $(\mathbf{\Phi}^2(\vec{\mathbf{r}}))^2$ local field theory.

The question of *boundary effects* in relation to the value of the configuration exponent γ also requires some study [13]. It caused some confusion in earlier publications [11,12,13]. In part **I** of these lectures, we showed that a finite self-avoiding patch embedded in an infinite Gaussian manifold has exponent $\gamma=1$ for any $0<D<2,\ D\neq 1$. Here the cases of closed or open manifolds with free boundaries will be considered.

3.2. Renormalizability to first order

The validity of RG methods and of scaling laws was first justified at leading order in ε through explicit resummations in [20], in close analogy to the procedure described in § 2.3 above for the δ-interaction impurity model. We shall not repeat all the arguments here, but comment on some significant results.

Let us consider the spatial correlation function $\big\langle[\vec{\mathbf{r}}(x)-\vec{\mathbf{r}}(0)]^2\big\rangle$. For a Gaussian (infinite) manifold it equals

$$\big\langle[\vec{\mathbf{r}}(x)-\vec{\mathbf{r}}(0)]^2\big\rangle_0 = d\,[-2G(x)] = d\,A_D|x|^{2-D} = d\frac{2}{S_D(2-D)}|x|^{2-D}. \qquad (3.2)$$

In the presence of self-avoidance, it is expected to scale as:

$$\left\langle [\vec{\mathbf{r}}(x) - \vec{\mathbf{r}}(0)]^2 \right\rangle \propto |x|^{2\nu}, \tag{3.3}$$

with a swelling exponent $\nu \geq \nu_0 = (2-D)/2$ for $d \leq d^*$. It can be directly evaluated by resummation of leading divergences [20]:

$$\left\langle [\vec{\mathbf{r}}(x) - \vec{\mathbf{r}}(0)]^2 \right\rangle = d \frac{2}{S_D(2-D)} |x|^{2-D} \left(1 + \frac{a}{\varepsilon} b_D |x|^{\varepsilon/2}\right)^{a_0/a}, \tag{3.4}$$

where b_D is simply the bare interaction parameter b conveniently dressed by coefficients

$$b_D = (2\pi A_D)^{-d/2} b = \left[4\pi / S_D(2-D)\right]^{-d/2} b,$$

and where a_0 and a are two universal coefficients [20]:

$$a_0 = \frac{S_D^2}{D} \frac{2-D}{2}, \quad a = S_D^2 \left(1 + \frac{1}{2-D} \frac{\Gamma^2(D/(2-D))}{\Gamma(2D/(2-D))}\right), \tag{3.5}$$

The scaling behavior (3.3) is then directly recovered from (3.4) in the large distance or strong self-avoidance limit $b|x|^{\varepsilon/2} \to +\infty$, with a value of the swelling exponent ν at first order in ε:

$$\nu = \frac{2-D}{2} + \frac{1}{2} \frac{a_0}{a} \frac{\varepsilon}{2}, \tag{3.6}$$

or explicitly:

$$\nu = \frac{2-D}{2} \left\{1 + \frac{\varepsilon}{2} \frac{1}{2D} \left[1 + \frac{1}{2-D} \frac{\Gamma^2(D/(2-D))}{\Gamma(2D/(2-D))}\right]^{-1}\right\}, \tag{3.7}$$

in agreement with the result (3.24) of part **I**.

Similarly, for a manifold of finite volume $\mathcal{V} = X^D$, one defines a dimensionless excluded volume parameter z, as in part **I** of these lectures, by

$$z = b_D X^{2D - (2-D)d/2} = (2\pi A_D)^{-d/2} b X^{\varepsilon/2}. \tag{3.8}$$

One finds an effective size of the membrane:

$$R^2 = \left\langle [\vec{\mathbf{r}}(X) - \vec{\mathbf{r}}(0)]^2 \right\rangle = \mathcal{X}_0(z, \varepsilon)\ d \frac{2}{S_D(2-D)} X^{2-D} \tag{3.9}$$

where $\mathcal{X}_0(z,\varepsilon)$ is the *swelling factor* with respect to the Gaussian size $\left\langle[\vec{\mathbf{r}}(X)-\vec{\mathbf{r}}(0)]^2\right\rangle_0$ (3.2), as introduced in part **I**, Eq. (3.1). The direct resummation of leading divergences to all perturbative orders gives [20]:

$$\mathcal{X}_0(z,\varepsilon)=\left(1+\frac{a}{\varepsilon}b_D X^{\varepsilon/2}\right)^{a_0/a}=\left(1+\frac{a}{\varepsilon}z\right)^{a_0/a}. \tag{3.10}$$

At first order in z, we recover

$$\mathcal{X}_0(z,\varepsilon)=1+\frac{a_0}{\varepsilon}z+\mathcal{O}(z^2), \tag{3.11}$$

which is the perturbative result (2.44) of part **I**.

We also intoduced in part **I**, Eqs. (3.7-9), the *dimensionless second virial coefficient* g

$$g=-(2\pi R^2/d)^{-d/2}\frac{\mathcal{Z}_{2,c}}{\mathcal{Z}_1^2}, \tag{3.12}$$

where $\mathcal{Z}_1$ and $\mathcal{Z}_{2,c}$ are respectively the (connected) 1-manifold and 2-manifold partition functions. The same direct resummation of leading divergences in perturbation theory gives for g

$$g=\frac{z}{1+az/\varepsilon}, \tag{3.13}$$

with a first order expansion

$$g=z-z^2a/\varepsilon+\cdots, \tag{3.14}$$

in agreement with **I.** Eqs. (3.16-17) [a was noted as a'_D there.]

It is interesting to observe the following fact, key to a rigorous approach to renormalizablity to first order. The RG flow equations were obtained in part **I.** Eqs. (3.4) (3.19) (3.22) from first order results (here **II.** (3.11), (3.14)) for the scaling functions

$$W(g,\varepsilon)=X\frac{\partial g}{\partial X}=\frac{\varepsilon}{2}z\frac{\partial g}{\partial z}=\frac{\varepsilon}{2}z-z^2a+\cdots=\frac{\varepsilon}{2}g-g^2\frac{a}{2}+\mathcal{O}(g^2) \tag{3.15}$$

$$X\frac{\partial}{\partial X}\ln\mathcal{X}_0(z,\varepsilon)=\frac{\varepsilon}{2}z\frac{\partial}{\partial z}\ln\mathcal{X}_0(z,\varepsilon)=\frac{1}{2}a_0\ z+\cdots=\frac{1}{2}a_0\ g+\mathcal{O}(g^2). \tag{3.16}$$

When truncated to this order, their solutions are exactly the resummed expressions (3.13) and (3.10). Turning things around, the direct resummation of leading poles in ε indeed establishes one-loop renormalizability [18,20].

3.3. **Renormalizabilty to all orders**

We briefly describe below the formalism that allows to prove the validity of the RG approach to self-avoiding manifolds, as well as to a larger class of manifold models with non-local interactions. (See [22], [26], for further details.). This formalism is based on an operator product expansion involving *multi-local singular operators*, which allows a systematic analysis of the short-distance ultraviolet singularities of the Edwards model. At the critical dimension $d^\star$, one can classify all of the relevant operators and show that the model (3.1) is *renormalizable to all orders* by renormalizations (i) of the coupling b, and (ii) of the position field $\vec{\mathbf{r}}$. As a consequence, one establishes the validity of scaling laws for *infinite* membranes, as well as the existence of finite size scaling laws for *finite* membranes. The latter result ensures the consistency of the DR approach.

A peculiar result, which distinguishes manifolds with non-integer D from open linear polymers with $D = 1$, is the absence of *boundary* operator multiplicative renormalization, leading to the general *hyperscaling relation* for the configuration exponent γ

$$\gamma = 1 - \nu d \,, \tag{3.17}$$

valid for finite SAM with $0 < D < 2$, $D \neq 1$. Note that this hyperscaling value is also valid for *closed* linear polymers (see, *e.g.*, [31].) This result is valid for closed or open manifolds with free boundaries, and has the same origin as the result $\gamma = 1$ obtained in part **I** for a finite SA patch embedded in an infinite manifold (see **I.** § 2.2.2 and § 2.2.3.)

3.4. **Perturbation theory and dipole representation**

As in part **I**, the partition function is defined by the functional integral:

$$\mathcal{Z} \;=\; \int \mathcal{D}[\vec{\mathbf{r}}(x)] \, \exp\left(-\mathcal{H}[\vec{\mathbf{r}}]/\mathrm{k_B}T\right) \;. \tag{3.18}$$

It has a perturbative expansion in b, formally given by expanding the exponential of the contact interaction

$$\begin{aligned} \mathcal{Z} \;&=\; \mathcal{Z}_0 \, \sum_{N=0}^{\infty} \frac{(-b/2)^N}{N!} \int \prod_{i=1}^{2N} d^D x_i \, \Big\langle \prod_{a=1}^{N} \delta^d(\vec{\mathbf{r}}(x_{2a}) - \vec{\mathbf{r}}(x_{2a-1})) \Big\rangle_0 \\ &\equiv\; \mathcal{Z}_0 \, \sum_{N=0}^{\infty} \frac{(-b/2)^N}{N!} \, Z_N \;, \end{aligned} \tag{3.19}$$

where $\mathcal{Z}_0$ is the partition function of the Gaussian manifold (hence $Z_0 \equiv 1$), and $\langle \cdots \rangle_0$ denotes the average with respect to the Gaussian manifold ($b = 0$):

$$\langle (\cdots) \rangle_0 \;=\; \frac{1}{\mathcal{Z}_0} \int \mathcal{D}[\vec{\mathbf{r}}(x)] \, \exp\left(-\frac{1}{2} \int d^D x \, (\nabla_x \vec{\mathbf{r}}(x))^2 \right) (\cdots) \; . \tag{3.20}$$

Physical observables are provided by average values of operators, which must be invariant under global translations. Using Fourier representation, local operators can always be generated by the exponential operators (or vertex operators), of the form

$$V_{\vec{\mathbf{q}}}(z) \;=\; \mathrm{e}^{\mathrm{i}\vec{\mathbf{q}}\cdot\vec{\mathbf{r}}(z)} \; . \tag{3.21}$$

In perturbation theory the field $\vec{\mathbf{r}}(x)$ will be treated as a massless free field and the momenta $\vec{\mathbf{q}}$ will appear as the "charges" associated with the translations in $\mathbb{R}^d$. Translationally invariant operators are then provided by "neutral" products of such local operators,

$$O_{\vec{\mathbf{q}}_1,\cdots,\vec{\mathbf{q}}_P}(z_1,\cdots,z_P) \;=\; \prod_{l=1}^{P} V_{\vec{\mathbf{q}}_l}(z_l) \; , \qquad\qquad \vec{\mathbf{q}}_{\mathrm{total}} \;=\; \sum_{l=1}^{P} \vec{\mathbf{q}}_l \;=\; \vec{\mathbf{0}} \; . \tag{3.22}$$

The perturbative expansion for these observables is simply

$$\begin{aligned} \Big\langle \prod_{l=1}^{P} \mathrm{e}^{\mathrm{i}\vec{\mathbf{q}}_l\cdot\vec{\mathbf{r}}(z_l)} \Big\rangle \;&=\; \frac{1}{\mathcal{Z}} \sum_{N=0}^{\infty} \frac{(-b/2)^N}{N!} \int \prod_{i=1}^{2N} d^D x_i \Big\langle \prod_{l=1}^{P} \mathrm{e}^{\mathrm{i}\vec{\mathbf{q}}_l\cdot\vec{\mathbf{r}}(z_l)} \prod_{a=1}^{N} \delta^d(\vec{\mathbf{r}}(x_{2a}) - \vec{\mathbf{r}}(x_{2a-1})) \Big\rangle_0 \\ &\equiv\; \frac{1}{\mathcal{Z}} \sum_{N=0}^{\infty} \frac{(-b/2)^N}{N!} \, Z_N\left(\{\vec{\mathbf{q}}_l\}\right) \; . \end{aligned} \tag{3.23}$$

Each δ function in (3.19) and (3.23) can itself be written in terms of two exponential operators as

$$\delta^d(\vec{\mathbf{r}}(x_2) - \vec{\mathbf{r}}(x_1)) \;=\; \int \frac{d^d\vec{\mathbf{k}}_1 d^d\vec{\mathbf{k}}_2}{(2\pi)^d} \, \delta^d(\vec{\mathbf{k}}_1 + \vec{\mathbf{k}}_2) \, \mathrm{e}^{\mathrm{i}\vec{\mathbf{k}}_1\cdot\vec{\mathbf{r}}(x_1)} \, \mathrm{e}^{\mathrm{i}\vec{\mathbf{k}}_2\cdot\vec{\mathbf{r}}(x_2)} \; . \tag{3.24}$$

Viewing again the momenta $\vec{\mathbf{k}}_1$, $\vec{\mathbf{k}}_2$ as charges assigned to the points x_1, x_2, the bi-local operator (3.24) corresponds to a dipole, with charges $\vec{\mathbf{k}}_1 = \vec{\mathbf{k}}$, $\vec{\mathbf{k}}_2 = -\vec{\mathbf{k}}$, integrated over its internal charge $\vec{\mathbf{k}}$. We depict graphically each such dipole as

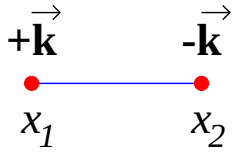

Fig. 5: *The dipole representing the δ interaction in (3.24).*

Similarly, the product of bi-local operators in (3.19) and (3.23) can be written as an ensemble of N dipoles, that is as the product of $2N$ vertex operators with N "dipolar constraints"

$$\mathcal{C}_a\{\vec{\mathbf{k}}_i\} \;=\; (2\pi)^d \delta^d(\vec{\mathbf{k}}_{2a-1} + \vec{\mathbf{k}}_{2a}) \;, \tag{3.25}$$

then integrated over all internal charges $\vec{\mathbf{k}}_i$:

$$\prod_{a=1}^{N} \delta^d(\vec{\mathbf{r}}(x_{2a}) - \vec{\mathbf{r}}(x_{2a-1})) = \int \prod_{i=1}^{2N} \frac{d^d\vec{\mathbf{k}}_i}{(2\pi)^d} \prod_{a=1}^{N} \mathcal{C}_a\{\vec{\mathbf{k}}_i\} \prod_{i=1}^{2N} \mathrm{e}^{\mathrm{i}\vec{\mathbf{k}}_i\cdot\vec{\mathbf{r}}(x_i)}. \tag{3.26}$$

Products of such bi-local operators and of external vertex operators, as in (3.23), are depicted by diagrams such as that of Fig. 6.

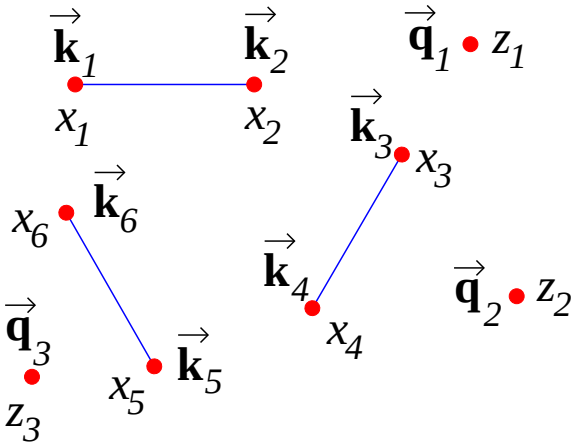

Fig. 6: *Dipole and charges representing bi-local operators and external vertex operators in (3.23).*

The Gaussian average in (3.19), (3.26) is easily performed, and with the neutrality condition $\sum_i \vec{\mathbf{k}}_i = \vec{\mathbf{0}}$, we can rewrite it as

$$\big\langle \prod_i \mathrm{e}^{\mathrm{i}\vec{\mathbf{k}}_i\cdot\vec{\mathbf{r}}(x_i)} \big\rangle_0 \;=\; \exp\Big(-\frac{1}{2}\sum_{i,j} \vec{\mathbf{k}}_i\cdot\vec{\mathbf{k}}_j G(x_i - x_j)\Big) \;, \tag{3.27}$$

with, as before, the translationally invariant two-point function

$$G(x_i - x_j) \;=\; -\frac{1}{2}\,\big\langle \big(\vec{\mathbf{r}}(x_i) - \vec{\mathbf{r}}(x_j)\big)^2 \big\rangle_0 \;=\; -\frac{|x_i - x_j|^{2-D}}{(2-D)S_D} \;. \tag{3.28}$$

Integration over the momenta $\vec{\mathbf{k}}_i$ then gives for the N'th term of the perturbative expansion for the partition function $\mathcal{Z}$ (3.19) the "manifold integral"

$$Z_N = (2\pi)^{-Nd/2} \int \prod_{i=1}^{2N} d^D x_i \; \Delta\{x_i\}^{-\frac{d}{2}} \;, \tag{3.29}$$

with $\Delta\{x_i\}$ the determinant associated with the auxiliary quadratic form (now on $\mathbb{R}^{2N}$) $Q\{k_i\} \equiv \sum_{i,j=1}^{2N} k_i k_j\, G(x_i, x_j)$ restricted to the N-dimensional vector space defined by the N neutrality constraints $\mathcal{C}_a\{k_i\}$, $k_{2a}+k_{2a-1}=0$. $\Delta\{x_i\}$ is given explicitly by the determinant of the $N \times N$ matrix Δ_{ab} (with row and columns labeled by the dipole indices $a, b = 1, \cdots, N$)

$$\Delta \;=\; \det\big(\Delta_{ab}\big), \quad \Delta_{ab} \;=\; G(x_{2a-1}, x_{2b-1}) + G(x_{2a}, x_{2b}) - G(x_{2a-1}, x_{2b}) - G(x_{2a}, x_{2b-1}). \tag{3.30}$$

Similarly, the N'th term in the perturbative expansion of the P-point observable (3.23) is

$$Z_N\left(\{\vec{\mathbf{q}}_l\}\right) = (2\pi)^{-Nd/2} \int \prod_{i=1}^{2N} d^D x_i\; \Delta\{x_i\}^{-\frac{d}{2}}\; \exp\left(-\frac{1}{2} \sum_{l,m=1}^{P} \vec{\mathbf{q}}_l \cdot \vec{\mathbf{q}}_m\, \frac{\Delta^{lm}}{\Delta}\right). \tag{3.31}$$

Δ^{lm} is the (lm) minor of the $(P+N)\times(P+N)$ matrix

$$\begin{bmatrix} G(z_l, z_m) & G(z_l, x_{2b-1}) - G(z_l, x_{2b}) \\ G(x_{2a-1}, z_m) - G(x_{2a}, z_m) & \Delta_{ab} \end{bmatrix}_{\substack{1\le l,m\le P \\ 1\le a,b\le N}}. \tag{3.32}$$

Note that a proper analytic continuation in D of (3.29) and (3.31) is insured, as in Section **2** above, by the use of distance geometry, where the Euclidean measure over the x_i is understood as the corresponding measure over the mutual squared distances $a_{ij} = |x_i - x_j|^2$, a distribution analytic in D [22].

3.5. **Singular configurations and electrostatics in $\mathbb{R}^D$**

The integrand in (3.29) is singular when the determinant vanishes, $\Delta\{x_i\} = 0$, or undefined if the latter becomes negative. The associated quadratic form $Q\{k_i\} = \sum_{i,j=1}^{2N} k_i k_j\, G(x_i, x_j)$, restricted by the N neutrality constraints $\mathcal{C}_a\{k_i\}$: $k_{2a}+k_{2a-1}=0, a = 1, \cdots, N$, is exactly the *electrostatic energy* of a gas of $2N$ scalar charges k_i located at points x_i in $\mathbb{R}^D$, and constrained to form N neutral pairs a of charges (dipoles). For such a globally neutral gas, the Coulomb energy is *minimal* when the charge density is *zero everywhere*, *i.e.*, when the non zero charges k_i aggregate into *neutral* "atoms". When $0 < D < 2$, because of the vanishing of the Coulomb potential at the origin, $G(0) = 0$, the corresponding minimal energy is furthermore *zero*, which implies that the quadratic form Q is *non-negative*, and thus its determinant is also non-negative: $\Delta \geq 0$.

Singular $\{x_i\}$ configurations, with $\Delta = 0$, still exist when Q is degenerate, which happens when some dipoles are assembled in such a way that, with appropriate non-zero

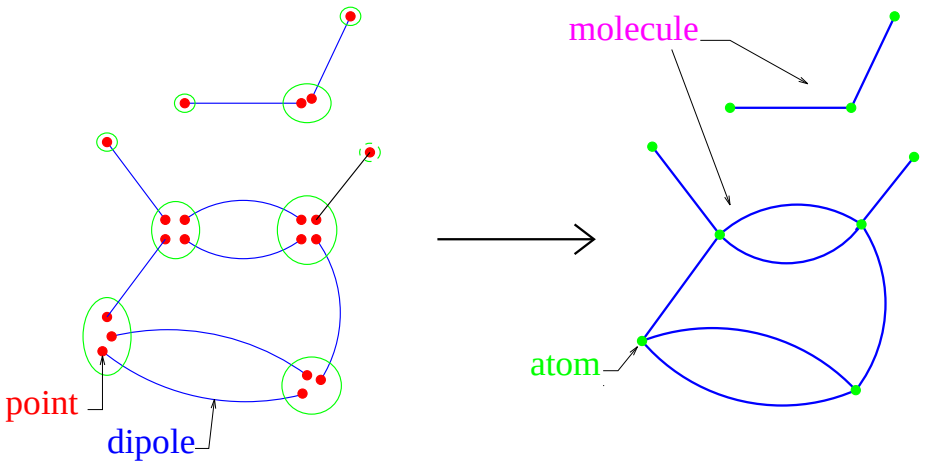

Fig. 7: *The notions of "atoms" and "molecules", built up from dipoles.*

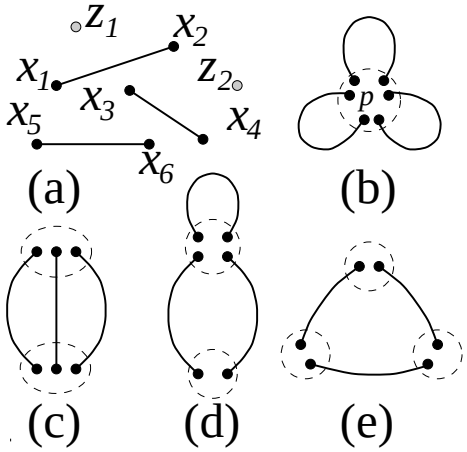

Fig. 8: *A general diagram with two external points and three internal dipoles, representing bi-local interactions δ_a (a); "molecules" describing singular configurations with one (b), two (c,d) and three (e) "atoms". (b,c,d) give UV divergences, (e) does not.*

charges, they still can build neutral atoms. This requires some of the points x_i to coincide *and* the corresponding dipoles to form at least one closed loop (Fig. 7). This ensures that the only sources of divergences are *short-distance singularities*, and extends the Schoenberg theorem used above.

3.6. Multi-local Operator Product Expansion

A singular configuration can thus be viewed as a connected "molecule" (Fig. 7), characterized by a set $\mathcal{M}$ of "atoms" p with assigned positions x_p, and by a set $\mathcal{L}$ of links a between these atoms, representing the dipolar constraints $\mathcal{C}_a$ associated with the

$\delta_a \equiv \delta^d(\vec{\mathbf{r}}(x_{2a}) - \vec{\mathbf{r}}(x_{2a-1}))$ interactions. For each p, we denote by $\mathcal{P}_p$ the set of charges i, at x_i, close to point p, which build the atom p and define the relative (short) distances $y_i = x_i - x_p$ for $i \in \mathcal{P}_p$ (Fig. 8).

The short-distance singularity of $\Delta^{-d/2}$ is then analyzed by performing a small y_i expansion of the product of the bilocal operators δ_a for the links $a \in \mathcal{L}$, in the Gaussian manifold theory (Eq. (3.23)). This expansion around $\mathcal{M}$ can be written as a *multi-local operator product expansion* (MOPE)

$$\prod_{a\in\mathcal{L}} \delta^d(\vec{\mathbf{r}}(x_{2a}) - \vec{\mathbf{r}}(x_{2a-1})) = \sum_{\Phi} \Phi\{x_p\}\, C^{\Phi}_{\underbrace{\delta\ldots\delta}_{|\mathcal{L}|}}\{y_i\} \tag{3.33}$$

where the sum runs over all multi-local operators Φ of the form:

$$\Phi\{x_p\} = \int d^d\vec{\mathbf{r}} \prod_{p\in\mathcal{M}} \Big\{ :\big\{(\nabla_{\vec{\mathbf{r}}})^{q_p}\, \delta^d(\vec{\mathbf{r}} - \vec{\mathbf{r}}(x_p))\big\}\; A_p(x_p):\Big\} \tag{3.34}$$

Here $A_p(x_p) \equiv A^{(r_p,s_p)}\,[\nabla_x\,;\vec{\mathbf{r}}(x_p)]$ is a local operator at point x_p, which is a combination of powers of x-derivatives and field $\vec{\mathbf{r}}$, of degree s_p in $\vec{\mathbf{r}}(x_p)$ and degree $r_p \geq s_p$ in ∇_x. $(\nabla_{\vec{\mathbf{r}}})^{q_p}$ denotes a product of q_p derivatives with respect to $\vec{\mathbf{r}}$, acting on $\delta^d(\vec{\mathbf{r}} - \vec{\mathbf{r}}(x_p))$. The symbol "$:\ \ :$" denotes the *normal product* subtraction prescription at x_p (which, in a Gaussian average, amounts to setting to zero any derivative of the propagator G_{ij} at coinciding points $x_i = x_j = x_p$). For $\mathrm{Card}(\mathcal{M}) \equiv |\mathcal{M}| > 1$, (3.34) describes the most general $|\mathcal{M}|$-body contact interaction between the points x_p, with possible inserted local operators $A_p(x_p)$ at each point x_p. For $|\mathcal{M}| = 1$, it reduces to a local operator $A_p(x_p)$.

The coefficient associated with the operator Φ in the MOPE, $C^{\Phi}_{\delta\ldots\delta}\{y_i\}$, can be written as an integral over the momenta $\vec{\mathbf{k}}_i$:

$$C^{\Phi}_{\delta\ldots\delta}\{y_i\} = \int \prod'_{a\in\mathcal{L}} \mathcal{C}_a\{\vec{\mathbf{k}}_i\} \prod_{p\in\mathcal{M}} \left\{ \prod_{i\in\mathcal{P}_p} d^d\vec{\mathbf{k}}_i\; \big\{(\nabla_{\vec{\mathbf{k}}})^{q_p}\delta^d\big(\sum_{i\in\mathcal{P}_p}\vec{\mathbf{k}}_i\big)\big\}\; C^{A_p}\{y_i,\vec{\mathbf{k}}_i\}\, \mathrm{e}^{-\frac{1}{2}\sum_{i,j\in\mathcal{P}_p}\vec{\mathbf{k}}_i.\vec{\mathbf{k}}_j G_{ij}} \right\} \tag{3.35}$$

where $C^{A_p}\{y_i,\vec{\mathbf{k}}_i\}$ is a monomial in the $\{y_i,\vec{\mathbf{k}}_i\}$'s, associated with the operator A_p, of similar global degree r_p in the $\{y_i\}$'s, and s_p in the $\{\vec{\mathbf{k}}_i\}$'s. The product Π' is over all constraints $a \in \mathcal{L}$ but one.

The MOPE (3.33) follows from the expression (3.24) in terms of free field exponentials plus constraints, and is established in [26].

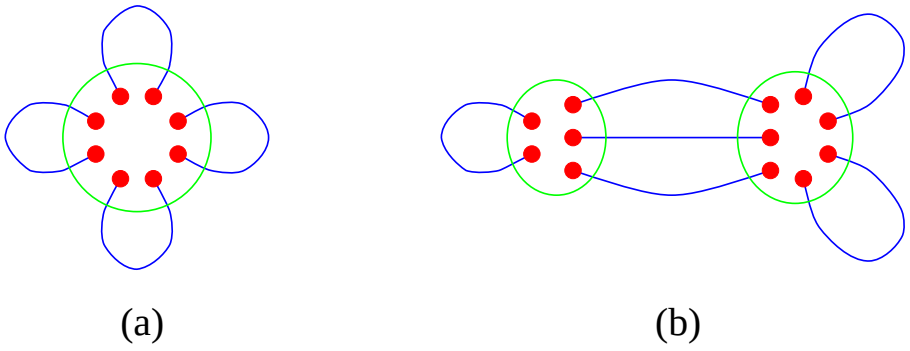

Fig. 9: *"Molecules" $\mathcal{M}$ producing (a) the one-body local elastic term $\Phi = :(\nabla\vec{\mathbf{r}}_p)^2:$, and (b) the two-body SA interaction term $\Phi = \delta^d(\vec{\mathbf{r}}_p - \vec{\mathbf{r}}_{p'})$.*

3.7. **Power counting and renormalization**

The MOPE (3.33) allows us to determine those singular configurations which give rise to actual UV divergences in the manifold integrals (3.29) or (3.31). Indeed, for a given singular configuration $\mathcal{M}$, by integrating over the domain where the relative positions $y_i = x_i - x_p$ are of order $|y_i| \lesssim \rho$, we can use the MOPE (3.33) to obtain an expansion of the integrand in (3.23) in powers of ρ. Each coefficient $C^{\Phi}_{\delta\dots\delta}$ gives a contribution of order ρ^{ω_Φ}, with degree ω_Φ given by power counting as

$$\omega_\Phi = D\{2|\mathcal{L}| - |\mathcal{M}|\} + d\nu_0\{|\mathcal{M}| - |\mathcal{L}| - 1\} + \sum_{p\in\mathcal{M}} \big\{\nu_0(q_p - s_p) + r_p\big\} \tag{3.36}$$

with $\nu_0 = (2-D)/2 < 1$ and $r_p \geq s_p$. Whenever $\omega_\Phi \leq 0$, a UV divergence occurs, as a factor multiplying the insertion of the corresponding operator Φ.

At the upper critical dimension $d^\star = 2D/\nu_0$, ω_Φ becomes *independent* of the number $|\mathcal{L}|$ of dipoles, and is equal to the canonical dimension ω_Φ of $\int \prod\limits_{p\in\mathcal{M}} d^D x_p\, \Phi\{x_p\}$ in the Gaussian theory.

Only three relevant multi-local operators Φ, with $\omega_\Phi \leq 0$ and such that the corresponding coefficient does not vanish by symmetry, are found by simple inspection. Two of these operators are *marginal* ($\omega_\Phi = 0$) at $d^\star$: (i) the one-body local elastic term $:(\nabla\vec{\mathbf{r}}_p)^2:$, obtained for $|\mathcal{M}| = 1$ ($q{=}0$, $r{=}s{=}2$); (ii) the two-body SA interaction term $\delta^d(\vec{\mathbf{r}}_p - \vec{\mathbf{r}}_{p'})$ itself, obtained through singular configurations with $|\mathcal{M}| = 2$ atoms (and with $q{=}r{=}s{=}0$ for p and p') (see Fig. 9).

A third operator is *relevant* with $\omega_\Phi = -D$, *i.e.*, the identity operator $\mathbf{1}$ obtained when $|\mathcal{M}| = 1$ ($q = r = s = 0$). It describes insertions of local "free energy" divergences

along the manifold, proportional to the manifold volume, which factor out of partition functions like (3.19), to cancel out in correlation functions (3.23), as already explained in part **I**, § 2.22.

The above analysis deals with *superficial UV divergences* only. A complete analysis of the general UV singularities associated with successive contractions toward "nested" singular configurations can be performed [26], using the same techniques as in [22] (**II.** § 2.8 above). A basic fact is that an iteration of the MOPE only generates multi-local operators of the same type (3.34).

The results are [26]: (i) that the observables (3.23) are UV finite for $d < d^\star(D)$, and are meromorphic functions in d with poles at $d = d^\star$; (ii) that a renormalization operation $\mathbf{R}$, similar to the subtraction operation of § 2.8 above [22], can be achieved to remove these poles; (iii) that this operation amounts to a renormalization of the Hamiltonian (3.1).

More explicitly, the renormalized correlation functions $\Big\langle \prod\limits_{l=1}^{P} \mathrm{e}^{\mathrm{i}\vec{\mathbf{q}}_l.\vec{\mathbf{r}}_{\mathbf{R}}(z_l)} \Big\rangle_{\mathbf{R}}$ have a finite perturbative expansion in the renormalized coupling $b_{\mathbf{R}}$, when $\big\langle \cdots \big\rangle_{\mathbf{R}}$ is the average w.r.t the renormalized Hamiltonian

$$\mathcal{H}_{\mathbf{R}}/\mathrm{k_B}T \;=\; \frac{1}{2}\,Z\int d^Dx\,\big(\nabla_x\vec{\mathbf{r}}_{\mathbf{R}}(x)\big)^2 + \frac{1}{2}\,b_{\mathbf{R}}\mu^{\varepsilon/2}Z_b\int d^Dx\int d^Dx'\;\delta^d\big(\vec{\mathbf{r}}_{\mathbf{R}}(x)-\vec{\mathbf{r}}_{\mathbf{R}}(x')\big)\;. \tag{3.37}$$

Here μ is a renormalization (internal) momentum scale, necessary for infinite manifolds, $\varepsilon = 4D - 2d\nu_0$; $Z(b_{\mathbf{R}})$ and $Z_b(b_{\mathbf{R}})$ are respectively the field and coupling constant renormalization factors, singular at $\varepsilon = 0$.

At first order, one finds by explicitly calculating $C_\delta^{(\nabla\vec{\mathbf{r}})^2}$ and $C_{\delta\delta}^{\delta}$ that [26]

$$Z = 1 + (2\pi A_D)^{-d/2}\frac{b_{\mathbf{R}}}{\varepsilon}\frac{S_D^2(2-D)}{2D},$$

$$Z_b = 1 + (2\pi A_D)^{-d/2}\frac{b_{\mathbf{R}}}{\varepsilon}\,\frac{S_D^2}{2-D}\,\frac{\Gamma^2(D/(2-D))}{\Gamma(2D/(2-D))},$$

with $A_D = [S_D(2-D)/2]^{-1}$. For quantities which do not stay finite in the infinite manifold limit $\mathcal{V}\to+\infty$, like partition functions, a shift in the free energy (*i.e.*, an "additive conterterm" in $\mathcal{H}_{\mathbf{R}}$), proportional to $\mathcal{V}$, is also necessary.

Expressing the observables of the SAM model (3.1) in terms of renormalized variables $\vec{\mathbf{r}} = Z^{1/2}\;\vec{\mathbf{r}}_{\mathbf{R}}$, $b = b_{\mathbf{R}}\;\mu^{\varepsilon/2}Z_b\;Z^{d/2}$, one can derive in the standard way RG equations involving Wilson's functions $W(b_{\mathbf{R}}) = \mu\frac{\partial}{\partial\mu}b_{\mathbf{R}}\big|_b$, $\nu(b_{\mathbf{R}}) = \nu_0 - \frac{1}{2}\mu\frac{\partial}{\partial\mu}\ln Z\big|_b$. A non-trivial IR fixed point $b_{\mathbf{R}}^\star \propto \varepsilon$ such that $W(b_{\mathbf{R}}^\star) = 0$ is found for $\varepsilon > 0$. It governs the large distance

behavior of the SA infinite manifold, which obeys scaling laws characterized by the size exponent ν. The value obtained in this approach, $\nu = \nu(b^\star_{\mathbf{R}})$, coincides with that obtained at first order in ε in Eq. (3.7) above [11-13, 20].[4]

3.8. **Finite size scaling and direct renormalization**

The direct renormalization formalism considered in part **I**, and in **II.** § 3.2 above, deals with *finite* manifolds with internal volume $\mathcal{V}$, and expresses scaling functions in terms of a dimensionless second virial coefficient (3.12) $g = -(2\pi R^2/d)^{-d/2}\mathcal{Z}_{2,c}/(\mathcal{Z}_1)^2$, where $\mathcal{Z}_1(\mathcal{V})(= \mathcal{Z}/\mathcal{Z}_0)$ and $\mathcal{Z}_{2,c}(\mathcal{V})$ are respectively the one- and (connected) two-membrane partition functions, and R is the effective radius of the membrane.

When dealing with a finite closed manifold (for instance the D-dimensional sphere $\mathcal{S}_D$ (**II.** § 2.6 and [22]), characterized by its (curved) internal metric, the massless propagator G gets modified. Nevertheless, from the short-distance expansion of G in a general metric,[5] one can show that the short-distance MOPE (3.33) remains valid. The expansion then extends to multi-local operators Φ of the form (3.34), with local operators $A(x)$ which may involve the Riemann curvature tensor and its derivatives, with appropriate coefficients $C^{\Phi}_{\delta\ldots\delta}$ [26]. Still, the coefficients for those operators Φ that do not involve derivatives of the metric stay the same as in Euclidean flat space.

At $d^\star$, UV divergences still come with insertions of relevant multi-local operators with $\omega_\Phi \leq 0$. When $0 < D < 2$, the operators involving curvature are found by power counting to be *all irrelevant*. Thus the *flat infinite membrane* counterterms Z and Z_b still renormalize the (curved) finite membrane theory. Standard arguments parallel to those of [16] for polymers then help to establish the direct renormalization formalism (see [26]). The second virial coefficient $g(b, \mathcal{V})$ (as any *dimensionless* scaling function) is UV finite once expressed as a function $g_{\mathbf{R}}(b_{\mathbf{R}}, \mathcal{V}\mu^D)$ of $b_{\mathbf{R}}$ (and μ). Then the scaling functions, when expressed in terms of g, obey RG flow equations, and stay *finite up to* $\varepsilon = 0$. The existence of a non-trivial IR fixed point $b^\star_{\mathbf{R}}$ for $\varepsilon > 0$ implies that in the large volume or strong interaction limit, $b\ \mathcal{V}^{\ \varepsilon/2D} \to +\infty$, g reaches a finite limit $g^\star = g_{\mathbf{R}}(b^\star_{\mathbf{R}})$ (independent of $\mathcal{V}\mu^D$), and so do all scaling functions. This is just direct renormalization, **QED**.

---

[4] On can notice the identity between coefficients in the renormalization factors Z, Z_b above, and (3.5).

[5] The expansion at the origin of the massless propagator $\tilde{G}$ on a curved manifold reads in Riemann normal coordinates $\tilde{G}(x) \simeq G(x) - \frac{|x|^2}{2D}\big\langle :(\nabla \mathbf{r})^2: \big\rangle$, with $G(x) \propto |x|^{2-D}$ the propagator in infinite flat space, and next order terms $\mathcal{O}(|x|^{4-D})$ proportional to the curvature and subdominant for $D < 2$; the normal product $(:\ \ :)$ is still defined w.r.t. infinite flat space, and gives explicitly for a finite manifold with volume $\mathcal{V}$ $\big\langle :(\nabla \mathbf{r})^2: \big\rangle = -1/\mathcal{V}$.

3.9. **Hyperscaling**

Let us first consider a *closed* manifold. As mentioned above, the renormalization of partition functions for a (finite) SAM requires a shift $(-f_D X^D)$ of the free energy, proportional to the manifold volume $\mathcal{V}$, and corresponding to the integration of a local contact divergence in the bulk. The configuration exponent γ is then defined by the scaling of the partition function[6]

$$\mathcal{Z}_1(\mathcal{V}) = \mathcal{Z}_0^{-1} \int \mathcal{D}[\vec{\mathbf{r}}]\, \delta^d(\vec{\mathbf{r}}(0))\, \mathrm{e}^{-\beta\mathcal{H} - f_D \mathcal{V}} \sim \mathcal{V}^{\frac{\gamma-1}{D}} \ . \tag{3.38}$$

A consequence of the absence for closed SAM, for $0 < D < 2$, of relevant geometrical operators other than the point insertion one, is the general hyperscaling law (3.17) relating γ to ν: $\gamma - 1 = -\nu d/D$. Indeed, from (3.38), $\mathcal{Z}_1$ is simply multiplicatively renormalized as $\mathcal{Z}_1(b, \mathcal{V}) = Z^{-d/2} \mathcal{Z}_1^{\mathbf{R}}(b_{\mathbf{R}}, \mathcal{V}\mu^D)$. This validates the hyperscaling hypothesis that $\mathcal{Z}_1 \sim \langle |\vec{\mathbf{r}}|^{-d} \rangle \sim \mathcal{V}^{-\nu d/D}$. Eq. (3.17) can been checked explicitly at order ε for the sphere $\mathcal{S}_D$ and the torus $\mathcal{T}_D$.

For an *open* SAM with *free* boundaries, and when $1 \leq D < 2$, the boundary operator $\int_{\text{boundary}} d^{D-1}x\, \mathbf{1}$ is *relevant*, requiring a boundary free energy shift $(-f_{D-1} X^{D-1})$. Since this does not enter into the bulk MOPE, it does not modify the renormalizations of $\vec{\mathbf{r}}$ and b. Furthermore, only for integer $D = 1$ is it *marginally relevant* [13], as explained in part **I**; thus for $D \neq 1$ the hyperscaling relation (3.17) *remains valid*. Only for open polymers at $D = 1$, do the corresponding (zero-dimensional) end-point divergences enter the multiplicative renormalization of $\mathcal{Z}_1$, and γ becomes an independent exponent. In polymer theory, an independent exponent actually appears for each *star vertex* [32].

Previous calculations [11,12,13] did not involve the massless propagator $\tilde{G}$ on a finite manifold with Neumann boundary conditions, but the simpler propagator G (3.28), corresponding to a finite SA patch immersed in an infinite Gaussian manifold. The same non-renormalization argument, as explained in [13] and in part **I**, yields $\gamma = 1$ for non-integer D.

When $D = 2$, operators involving curvature and boundaries become relevant, and (3.17) is not expected to hold, either for closed or open manifolds.

---

[6] Here we consider $\mathcal{Z}_1(\mathcal{V}) \equiv \exp(-f_D X^D)\ \mathcal{Z}/\mathcal{Z}_0$, *i.e.*, the *dimensionally regularized* partition function, which includes the free energy shift (see **I**. § 2.2.2).

3.10. Θ-point and long-range interactions

The above formalism is actually directly applicable to a large class of manifold models where the interaction can be expressed in terms of free field exponentials with suitable neutrality constraints $\mathcal{C}_a\{\vec{\mathbf{k}}_i\}$. Examples of such interactions are the n-body contact potentials, or the two-body long-range Coulomb potential $1/|\vec{\mathbf{r}}-\vec{\mathbf{r}}'|^{d-2}$, represented by modified dipolar constraints $\mathcal{C}\{\vec{\mathbf{k}}_i\} = |\vec{\mathbf{k}}|^{-2}\delta^d(\vec{\mathbf{k}}+\vec{\mathbf{k}}')$. In these models the MOPE involves the same multi-local operators as in (3.34), with coefficients (3.35) built with the corresponding constraints $\mathcal{C}_a$.

As an application of the MOPE, one finds that for a polymerized membrane at the Θ-point where the two-body term b in (3.1) vanishes, the most relevant short-range interaction is either the usual tricritical *three*-body contact potential, with u.c.d. $d_3^\star = 3D/(2-D)$, as for ordinary polymers [33], or the two-body singular potential $\Delta_{\vec{\mathbf{r}}}\delta^d(\vec{\mathbf{r}}-\vec{\mathbf{r}}')$ with u.c.d. $\tilde{d}_2^\star = 2(3D-2)/(2-D)$. The latter is the most relevant one when $D > 4/3$ (see [34]).

The very absence of *long-range* interactions in the MOPE shows that those interactions are not renormalized. When considering charged polymerized membranes with a two-body Coulomb potential for instance, the only (marginally) relevant operator at the upper critical dimension is the local elastic energy density $:(\nabla\vec{\mathbf{r}})^2:$, which indicates that only $\vec{\mathbf{r}}$ is renormalized. As a consequence, one can show that $\nu = 2D/(d-2)$ exactly, generalizing a well-known result for polymers [35].

We did not address here other interesting issues: the approach to the physical $D = 2$ case from the $D < 2$ manifold theory [27,36], numerical simulations of 2D polymerized membranes [10], or the question of the actual physical phase (crumpled or flat) of a two-dimensional polymerized membrane in d-space [37]. We have concentrated instead on those more fundamental aspects of renormalization theory, that have been driven by the fascinating properties of these fluctuating polymerized membranes.

Acknowledgements

These notes rely heavily on [18], and on the articles [21], [22]: *Renormalization Theory for Interacting Crumpled Manifolds*, by François David, B.D., and Emmanuel Guitter. For the self-avoiding manifold Edwards model, I followed [20], and [25], [26]: *Renormalization Theory for Self-Avoiding Polymerized Membranes*, by the same authors. I also wish to thank Emmanuel Guitter for his valuable help with the figures, and Thomas C. Halsey for a careful reading of the manuscript.

References

[1] *Statistical Mechanics of Membranes and Surfaces*, Proceedings of the Fifth Jerusalem Winter School for Theoretical Physics (1987), D. R. Nelson, T. Piran, and S. Weinberg Eds., World Scientific, Singapore (1989).

[2] S. F. Edwards, Proc. Phys. Soc. Lond. **85** (1965) 613.

[3] J. des Cloizeaux and G. Jannink, *Polymers in Solution, their Modelling and Structure*, Clarendon Press, Oxford (1990).

[4] X. Wen, C.W. Garland, T. Hwa, M. Kardar, E. Kokufuta, Y. Li, M. Orkisz, and T. Tanaka, Nature **355** (1992) 426.

[5] M.S. Spector, E. Naranjo, S. Chiruvolu, and J.A. Zasadzinski, Phys. Rev. Lett. **73** (1994) 2867.

[6] C.F. Schmidt, K. Svoboda, N. Lei, I.B. Petsche, L.E. Berman, C. Safinya, and G. S. Grest, Science **259** (1993) 952.

[7] S.I. Stupp, S. Son, H.C. Lin, and L.S. Li, Science **259** (1993) 59.

[8] H. Rehage, B. Achenbach, and A. Kaplan, Ber. Bunsenges. Phys. Chem. **101** (1997) 1683.

[9] Y. Kantor, M. Kardar, and D. R. Nelson, Phys. Rev. Lett. **57** (1986) 791; Phys. Rev. **A 35** (1987) 3056.

[10] See, *e.g.*, the lectures by G. Gompper and D.M. Kroll in this volume, and the references therein.

[11] M. Kardar and D. R. Nelson, Phys. Rev. Lett. **58** (1987) 1289, 2280(E); Phys. Rev. **A 38** (1988) 966.

[12] J. A. Aronowitz and T. C. Lubensky, Europhys. Lett. **4** (1987) 395.

[13] B. Duplantier, Phys. Rev. Lett. **58** (1987) 2733.

[14] J. des Cloizeaux, J. Phys. France **42** (1981) 635.

[15] P.G. de Gennes, Phys. Lett. **A 38** (1972) 339.

[16] M. Benhamou and G. Mahoux, J. Phys. France **47** (1986) 559.

[17] N. N. Bogoliubov and O. S. Parasiuk, Acta Math. **97** (1957) 227;
K. Hepp, Commun. Math. Phys. **2** (1966) 301;
W. Zimmermann, Commun. Math. Phys. **15** (1969) 208.

[18] B. Duplantier, Phys. Rev. Lett. **62** (1989) 2337.

[19] See also M. Lässig and R. Lipowsky, Phys. Rev. Lett. **70** (1993) 1131.

[20] B. Duplantier, T. Hwa, and M. Kardar, Phys. Rev. Lett. **64** (1990) 2022.

[21] F. David, B. Duplantier, and E. Guitter, Phys. Rev. Lett. **70** (1993) 2205, hep-th/9212102.

[22] F. David, B. Duplantier, and E. Guitter, Nucl. Phys. **B 394** (1993) 555, cond-mat/9211038.

[23] M. Cassandro and P.K. Mitter, Nucl. Phys. **B 422** (1994) 634.

[24] P.K. Mitter and B. Scoppola, Commun. Math. Phys. **209** (2000) 207.
[25] F. David, B. Duplantier, and E. Guitter, Phys. Rev. Lett. **72** (1994) 311, cond-mat/9307059.
[26] F. David, B. Duplantier, and E. Guitter, *Renormalization Theory for Self-Avoiding Polymerized Membranes*, cond-mat/9701136.
[27] F. David and K. Wiese, Phys. Rev. Lett. **76** (1996) 4564; Nucl. Phys. **B 487** (1997) 529, cond-mat/9608022.
[28] F. David and K. Wiese, Nucl. Phys. **B 535** (1998) 555, cond-mat/9807160.
[29] L. M. Blumenthal, *Theory and Applications of Distance Geometry*, Clarendon Press, Oxford (1953).
[30] M. C. Bergère and Y.-M. P. Lam, J. Math. Phys. **17** (1976) 1546.
[31] B. Duplantier, Nucl. Phys. **B 430** (1995) 489.
[32] B. Duplantier, J. Stat. Phys. **54** (1989) 581.
[33] B. Duplantier, J. Physique **43** (1982) 991.
[34] K. Wiese and F. David, Nucl. Phys. **B 450** (1995) 495, cond-mat/9503126.
[35] P. Pfeuty, R. M. Velasco, P.-G. de Gennes, J. Physique **38** (1977) L5.
[36] T. Hwa, Phys. Rev. **A 41** (1990) 1751.
[37] One may consult, *e.g.*, K. Wiese, in *Phase Transitions and Critical Phenomena*, C. Domb and J.L. Lebowitz eds., vol. 19 (2001), and the new chapters of the present volume by M. Bowick, G. Gompper and D.M. Kroll, and L. Radzihovsky.